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STABILITY OF FLOWS OF A WEAKLY COMPRESSIBLE FLUID IN A PLANE PIPE OF LARGE, BUT FINITE LENGTH

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Investigation of the stability of fluid flows in plane pipes [1] is usually associated with the investigation of the behavior, in time, of an infinite periodic wave of the form $\phi(y) \exp i(kx - \omega t)$ where k is real. The relation between ω and k is found from the condition of existence of a nontrivial solution of a boundary value problem for $\phi(y)$ and is defined by a multivalued analytic function $k(\omega)$. It was shown in [1 and 2] that the function $k(\omega)$ has only one branch $k_1(\omega)$ giving real values of k when $\operatorname{Im}\omega > 0$. This branch corresponds to the perturbations propagating downstream. Earlier [3] the author computed the function $k_1(\omega)$ for real ω for the case of flows of an incompressible fluid at large Reynolds' numbers. It is easily seen that the behavior of $k_1(\omega)$ will not be greatly altered when the fluid is compressible, provided that its compressibility is sufficiently small.

The condition of instability of the flow in a pipe of large but finite length, can be reduced to the fact [3 and 4] that Eq.

$$\operatorname{Im}[k_1(\omega) - k_a(\omega)] = 0 \tag{1}$$

has solutions ω when $\operatorname{Im}\omega > 0$. The expression $k_a(\omega)$ in (1) will, for the time being, denote the branch of $k(\omega)$ defining the wave number of some perturbation propagating upstream. We shall show that in the case of weakly compressible flows with high Reynolds numbers the above condition of instability holds, provided that the branch corresponding to acoustic oscillations propagating upstream is taken as $k_a(\omega)$.

If, either the fluid is compressible or the pipe walls are elastic, then acoustic or Zhukovskii waves may be set up and propagate along it. Their wavelength will, for the given frequency, be inversely proportional to the compressibility of the fluid and the walls. When the wavelength becomes large, we can neglect the transverse velocity and pressure gradient components. Excess pressure at some cross section will be proportional to the excess of mass per unit length of the pipe, so that

$$i\omega p = ik_a\rho_0 a^2 \frac{1}{2}\int_{-1}^{1} u\,dy \tag{2}$$

where k_a and ω are the wave number and frequency of the given wave, ρ_0 is the density of the fluid, a is the velocity of propagation of the perturbations and u is the longitudinal component of the velocity perturbation. In deriving (2), we have assumed that $\omega/k_a \gg u$.

Function $u(y)$ satisfies Eq.

$$u''(y) = -i\omega R u + iR\frac{k_a^2 a^2}{\omega}\frac{1}{2}\int_{-1}^{1} u\,dy \tag{3}$$

where R is the Reynolds' number. Let us normalize $u(y)$ so, that

$$\frac{1}{2}\int_{-1}^{1} u\,dy = 1 \tag{4}$$

Then the solution of (3) satisfying the zero boundary conditions can be written as

$$u = \frac{a^2k_a^2}{\omega^2}\left[1 - \frac{\operatorname{ch}\sqrt{-i\omega R}\,y}{\operatorname{ch}\sqrt{-i\omega R}}\right] \tag{5}$$

Inserting (5) into (4), we obtain the following expression relating ω and k_a

$$\frac{a^2k_a^2}{\omega^2}\left[1 - \frac{1}{\sqrt{-i\omega R}}\operatorname{th}\sqrt{-i\omega R}\right] = 1 \tag{6}$$

Since acoustic waves decay in a viscous fluid (see Eq. (7) below), we find that when $\operatorname{Im}\omega > 0$, then the inequality $\operatorname{Im} k_a < 0$ should hold for the wave propagating upstream. We know [1 and 3] that the values of ω lying on the upper complex semi-plane for which $\operatorname{Im} k_1(\omega) < 0$, vary with increasing R in such a manner, that $\omega R \to \infty$ as $R \to \infty$. Therefore, at high values of R it is sufficient to consider Eqs. (1) and (6) only for those values of ω, for which $\omega R \gg 1$. Then (6) yields the following relation for a wave moving upstream

$$k_a = -a^{-1}\omega\left(1 + \frac{1}{2}\frac{e^{i\pi/4}}{\sqrt{\omega R}}\right) \tag{7}$$

where we take the arithmetic branch of the root, contained in $-\pi/2 < \arg\omega < 3\pi/2$.

When $a \to \infty$, ω is arbitrary and $\operatorname{Im}\omega > 0$, $k_a \to 0$ and approaches the point $k = 0$ from the lower semi-plane. Therefore, at sufficiently large values of a and R, Eq. (1) has roots ω in the upper semi-plane and the flow is unstable.

Eq. (1) used to establish the natural frequencies of perturbations in the pipe, assumes for an incompressible fluid the limit form

$$\operatorname{Im} k_1(\omega) = 0 \tag{8}$$

The latter coincides with the equation for the complex frequency of an infinite periodic wave. This equation together with the inequality $\operatorname{Im}\omega > 0$ is usually employed [1] as the criterion of instability of the flow in an infinite pipe. It should however be noted, that in the case of an incompressible fluid, the existence of the branch $k_a(\omega) \equiv 0$ is not sufficient to justify writing (1) in the form of (8) directly, without putting $a \to \infty$. This is caused by the fact that the G.I. Petrovskii condition postulated by the author in the derivation of (1) in [4], is not valid in the case of an incompressible fluid. (Petrovskii condition states, that $\operatorname{Im} k \neq 0$ if $\operatorname{Im}\omega$ is sufficiently large for all perturbations, and it ensures the correctness of the statement of the Cauchy's problem).

We also note that the solutions $\phi(y)$ of the Orr-Sommerfeld equation corresponding to the branch $k_a(\omega) \equiv 0$ cannot be assumed to be eigenfunctions, since they only need to satisfy the condition $\phi'(\pm 1) = 0$ on each wall, and the condition of impermeability $k\,\phi(\pm 1) = 0$ does not restrict ϕ in any way.

From (4) it follows that the perturbations corresponding to $k_a(\omega)$ are related to the rate of change of the flow of fluid, and the latter becomes independent of x when $a = \infty$.

In the case when the boundary conditions at the ends of the pipe exclude the possibility of altering the rate of flow or, when the acoustic wave and the wave corresponding to $k_1(\omega)$ cannot generate each other by reflection from the pipe ends (e.g. due to the difference in symmetry, since the corresponding stream functions in y are even and odd respectively), then the branch $k_a(\omega)$ needs not be considered and can be replaced in (1) with another branch of $k(\omega)$ corresponding to the perturbations moving upstream. Such an equation was studied in [3] for an incompressible fluid and we showed that at high Reynolds' numbers it has no solutions ω in the upper semi-plane, i.e. under the given conditions, flows with fixed rate of flow are not globally [4] unstable.

When the pipe is of infinite length and the Reynolds' number is sufficiently high, the instability of the flow is removable, so that if the initial perturbation is bounded in space, then for $t \to \infty$, the perturbations tend to zero at any fixed point [5]. Thus, ends of the pipe which may alter the rate of flow destabilize the flow of a weakly compressible fluid, and at high R this leads to instability. On the other hand, flows which maintain the constant rate, remain stable.

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